28/111 PAPER – III

Subgroup

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Th1 Union of two subgroups is a subgroup iff one of them is contained in the other.

Proof Let H, K be two subgroup of a group G and suppose $H \subset K$ then $H \cup K = K$ which is a subgroup of G.
Conversely, let H, K be two subgroups of G s.t. $H \cup K$ is also a subgroup of G. We show one of them must be contained in the other. Suppose it is not true i.e

$H \not\subseteq K, K \subseteq H$

Then $\exists\, x \in H$ s.t. $x \notin K$

$\exists\, y \in K$ s.t. $y \notin H$

Also then $x, y \in H \cup K$ and since $H \cup K$ is a subgroup, $xy \in H \cup K$

$\Rightarrow xy \in H$ or $xy \in K$.

If $xy \in H$, then as $x \in H$, $x^{-1}(xy) \in H$

$\Rightarrow y \in H$, which is not true.

Again, if $xy \in K$, then as $y \in K$
$(xy)y^{-1} \in K \Rightarrow x \in K$ which is not true i.e either way we land up with a contradiction.

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Hence our supposition that $H \not\subseteq K$ and $K \not\subseteq H$ is wrong.
Thus one of the two is contained in the other.

Definition. Let H be a subgroup of a group G. For $a, b \in G$, we say a is congruent to b mod H if $ab^{-1} \in H$.
In notational form we write
$a \equiv b \bmod H$.

Def Let H be a subgroup of G and let $a \in G$ be any element. Then
$Ha = \{ha \mid h \in H\}$ is called a right coset of H in G.